Some aspects of a generalized Wagner model.

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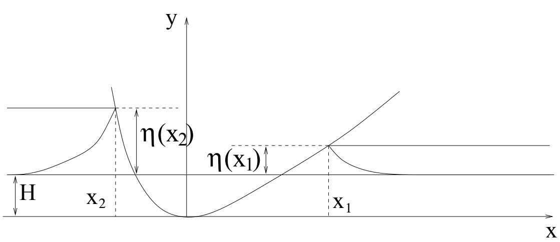
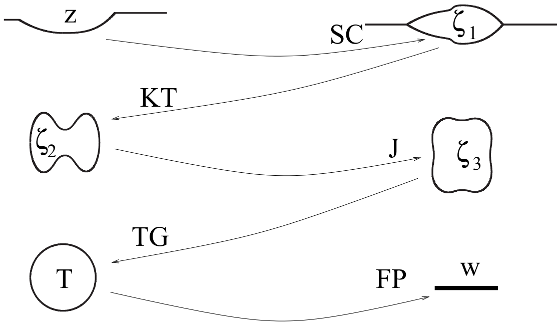
1. Introduction

Since the pionneering works by Wagner [1] hydrodynamic models for impact of bodies onto a flat free surface have been continuously developed. First Wagner proposed a model based on the flat disk approximation under the assumption that the deadrise angle is small; in the range [4 : 20] degrees. However, there are many situations where the linearization of the boundary condition on the wetted surface is not valid any longer. First attempts proposed by Zhao and Faltinsen [11],[10] or Battistin and Iafrati [1] (among others) solved the boundary value problem through a boundary integral equation method, keeping all the non linearities, including the free surface ones. Their models require significant computational resources. For a standard use in industry more friendly models can be developed. As an intermediate step, Mei *et al* [6] proposed a generalized Wagner model, formulating the boundary value problem (BVP) in potential theory, with no gravity and no surface tension. The impermeability condition is prescribed on the exact wetted surface. The originality of this technique comes from the fact that the dynamic free surface boundary condition reduces to a homogeneous Dirichlet condition for the potential on a horizontal line emanating from the contact points. Flow in the jet is not accounted for in this approach. On the basis of their works we consider here the impact of an asymmetric body. To this end, conformal mapping are implemented. The computational domain is bounded by the lower half space and the physical body contour is turned into a flat plate. We formulate a BVP in this domain. A method of resolution of this BVP is developed in the sequel and the mass conservation law is studied. The wetting corrections are computed following Mei's technique [6]. We give some applications concerning the symmetric wedge, for which analytical solution is available. Histories of wetting correction and slamming loads are computed for more general shapes as a bow ship form. We finally draw method of solution for more complicated shapes and for the fully hydroelastic coupling as well.

2. Transformation of the computational domain

The figure below shows the physical configuration. Conformal mappings are used to transform the fluid domain. The right figure sums up the successive mappings : Karmann-Trefftz (KT), Joukowsky (J), Theodorsen-Garrick (TG) and flate plate (FP). For sake of brievety, this part is not developed here.





3. Boundary Value Problem (BVP)

The conformal mapping leads to a BVP posed in the lower half space. It is described with the complex coordinate $w = u + iv$. It is bounded by the free surface $|u| > 1$ and a flat plate of unit half length $|u| < 1$. We introduce the complex potential F which can be formulated by using the theoretical works by Gahkov [2] (see Art. 42.3 pp 426–427, 1990).

$$F(w) = \phi + i\psi = -\frac{1}{\pi}\sqrt{1-w^2}\int_{-1}^{1}\frac{\psi(\tau)d\tau}{\sqrt{1-\tau^2}(\tau - w)}, \quad \int_{-1}^{1}\frac{\psi(\tau)d\tau}{\sqrt{1-\tau^2}} = 0. \tag{1}$$

The last condition stipulates that the analytic function F is bounded at $w = \pm 1$. The Neumann boundary condition on the body ($\phi_{,n} = \vec{U}\vec{n}$) is then transformed yielding a Dirichlet condition for the streamfunction ψ which is introduced in the integrand of (1). It is worth using the intermediate complex plane $T = re^{i\theta}$ where the body contour is a unit half circle $|T| = 1$. Then the horizontal coordinate along the physical body contour can be parametrized with the azimuthal coordinate θ and it is expressed as a Fourier series

$$x(\theta) = \sum_{n=0}^{\infty} A_n \cos(n\theta). \tag{2}$$

The vertical velocity on the free surface is obtained by differentiating (1) and it reads

$$\phi_{,y} = \frac{U}{J(u)} \sum_{n=1}^{\infty} A_n L_n(u), \tag{3}$$

where J is the Jacobian of the total transformation expressed on the free surface (it is real) and $L_n(u)$ are a set of real functions which can be calculated recursively and analytically.
From this result, we can examine the mass conservation law. We evaluate the fluid volume above and under the undisturbed mean water level. The following identity must thus be checked

$$H(x_1 - x_2) - \int_{x_2}^{x_1} f(x)dx = \int_{-\infty}^{x_2} \eta(x,t)dx + \int_{x_1}^{\infty} \eta(x,t)dx. \tag{4}$$

This identity is time differentiated yielding

$$U(x_1 - x_2) = \int_{-\infty}^{x_2} \eta_{,t}(x,t)dx + \int_{x_1}^{\infty} \eta_{,t}(x,t)dx. \tag{5}$$

Substituting $\eta_{,t}$ with $\phi_{,y}$ as given in equation (3), identity (5) is effectively checked.

4. Method of solution

To calculate the wetting corrections two equations must be solved. They follow from the time integration of the kinematic free surface condition written at the contact points (x_1, x_2).

$$f(x_j) - H(t) \;=\; \int_0^t \phi_{,y}(x_j(t), \eta(x_j(\tau),\tau),\tau)d\tau \quad j \;=\; 1..2. \tag{6}$$

$\phi_{,y}$ is given by (3) and it only depends on $(x_j(t), x_1(\tau), x_2(\tau))$. $H(t)$ is introduced in the integrand of (6) and a new function W is defined as

$$U(\tau) + \phi_{,y}(x_j(t), y_j(\tau), \tau) = U(\tau)W(x_j(t), x_1(\tau), x_2(\tau)). \tag{7}$$

Hence, the change of variable $\ell = x_j(\tau)$ for $\tau \leq t$, in (6) yields

$$f(x_1(t)) = \int_0^{x_1(t)} U(\ell)W(x_1(t), \ell, x_2(\ell))\frac{d\tau}{d\ell}d\ell. \tag{8}$$

Following Mei's method, $U(\ell)\frac{d\tau}{d\ell}$ is decomposed on a basis of Chebyshev's polynomials such as :

$$U(\ell)\frac{d\tau}{d\ell} = \sum_{j=0}^{N} a_j^{(1)} T_j(\ell) = \sum_{j=0}^{N} b_j^{(1)} \ell^j, \tag{9}$$

where $T_j(\ell)$ is the j^{th} Chebyshev polynomial of the first kind with a proper scaling. Knowing the coefficients $a_j^{(1)}$ or $b_j^{(1)}$, integration of (9) over $[0 : x_1(t)]$ provides the final value of $x_1(t)$. The first equation (8) is turned into

$$f(x_1(t)) = \sum_{j=0}^{N} a_j^{(1)} \int_0^{x_1(t)} W(x_1(t), \ell, x_2(\ell))T_j(\ell)d\ell. \tag{10}$$

Equation (10) is solved by collocation. The variable $x_1(t)$ is set to discrete values over the interval $[0 : X_1]$, say $X_1^{(k)}$ the k^{th} zeroes of the Chebyshev polynomial T_{N+1}.

The new set of equations for x_1, x_2 is highly nonlinear and we consider solving them with a fixed point algorithm provided that the functions which link x_2 to x_1 have "good" mathematical properties. To start the iterative process we need a first guess of $x_2(\ell)$ or $x_1(\ell)$. To this end we use the solution of the linearized Wagner model (see Scolan *et al* [7]). We know that the two Wagner conditions give

$$\int_{x_2}^{x_1} f(x)\sqrt{\frac{x_1 - x}{x - x_2}}\,dx = \int_{x_2}^{x_1} f(x)\sqrt{\frac{x - x_2}{x_1 - x}}\,dx. \tag{11}$$

Then we have a way to calculate x_2 for a given set of x_1 over the interval $[0 : X_1]$. The same is possible for $x_2 \in [0 : X_2]$. Equation (10) and its analog for x_2 are then solved yielding the coeficients $a_j^{(1)}$ and $a_j^{(2)}$. Hence we have information enough to recompute the new function $x_2(\ell)$ or $x_1(\ell)$. The process is pursued until convergence. When the coefficients $a_j^{(1)}$ and $a_j^{(2)}$ do not change any longer (for a given accuracy), the final wetting corrections are computed.

5. Applications

The asymmetric case has not been yet implemented. We give here preliminary results for symmetric bodies: wedge and bow ship form. In the former case, we can derive analytically the equation of the mass conservation law as well. For the latter case, we illustrate the method by comparing our results with the Modified Logvinovich Model (MLM) method (see [3] and [4]).

Symmetric wedge at constant velocity

In this particular case, time derivative of the free surface elevation could be analytically evaluated by using the Schwarz Christoffel (SC) transformation

$$\frac{x\,\tan(\alpha)}{\gamma\,U\,t} = \frac{1}{A}\int_0^{p(x,t)}\left(\frac{w^2}{w^2+1}\right)^{\beta} dw + 1, \tag{12}$$

where $p(x,t)$ is the image by SC of a point x on the free surface at time t, α is the deadrise angle and $\beta = \frac{1}{2} - \frac{\alpha}{\pi}$, A is a constant of normalization, used to set the vertical velocity of the flow to U at infinity and γ is defined as

$$\gamma - 1 = H(t)\,\eta(X(t), t). \tag{13}$$

Noting that $x_1 = x_2 = X(t)$, condition (5) reduces to

$$X(t) - \frac{1}{U}\int_{X(t)}^{\infty} (\eta_{,t}(x,t) - U)\,dx = 0. \tag{14}$$

The free surface elevation η is analytically known as :

$$\eta(x,t) = \frac{x\,\tan(\alpha)}{UtA\gamma}\int_{p(x,t)}^{\infty}\left(\int_0^u\left(\frac{w^2}{1+w^2}\right)^{\beta} dw + A\right)^{-2} du - 1. \tag{15}$$

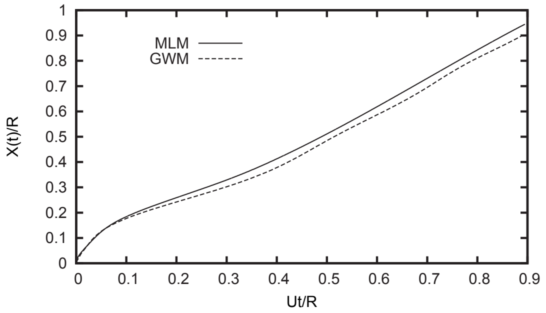
Its time derivative is :

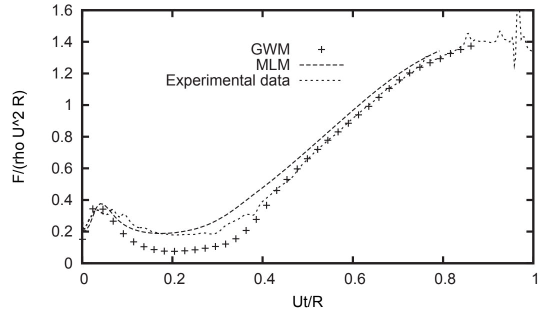
$$\eta_{,t}(x,t) = U\left(\frac{p(x,t)^2+1}{p(x,t)^2}\right)^{\beta} \tag{16}$$

Introducing (16) into (14) we show after some algebra that the mass conservation law is verified in effect for each value of α. Note that for a disymmetric wedge, an analytical solution also exists. In that case the integrand in the SC transformation (12) has the form : $\frac{w^{\alpha_1+\alpha_2}}{(w+1)^{\alpha_1}(w-1)^{\alpha_2}}$, where α_1 and α_2 are the deadrise angles on both sides. The same conclusions can be drawn regarding the mass conservation.

Bow ship section

The method developed above is used to study the bow ship form defined by Korobkin and Malenica in [4]. Figure below shows the time variation of the wetting correction. MLM results are compared to the present Generalized Wagner (GW) approach. Right figure shows the history of the force F acting on the entering body. It is made non dimensional with ρ the density of the fluid, R the half size of the section in the horizontal direction and U^2. The force is computed from numerical integration of the pressure. The pressure follows from Bernoulli's equation and contains all terms. In the present approach, we determine the point at which the pressure vanishes then bounding the support of integration. The largest error occurs when the deadrise angle is maximum. More insight into this behaviour is necessary.





6. Conclusion

The present approach draws new tracks to other problems. The first one is the treatment of the axisymmetric case. In fact it follows from the two-dimensional case straightfowardly. As conformal mappings cannot be used any longer, the flow is computed through an integral equation. Due to axisymmetry of the problem, this integral equation can be significantly simplified. The same method of solution as Mei's technique can be used. The second problem concerns more arbitrary three-dimensional shapes for which a Shorygin-like [8] method can be used. Quasi axisymmetric case is treated and the obtained results are compared to the theoretical ones elaborated by Korobkin and Scolan [5]. Finally, as a third step, the present approach is indicated either for two-dimensional or axisymmetric shapes to deal with a strong hydro-elastic coupling when large deformations of bodies are expected as for inflated floaters impacting water.
More details will be given in the conference concerning conformal mapping, practical use of the present method and more results as well.

7. References

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